

Homework 9

Math 121A, Fall 2025.

Due: Monday, October 27, 11:59PM (completion only)

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. Find all the singularities of the function $f(z) = 1/\sin(z)$, as well as the residues at those singularities. Compute the values of the following integrals:

$$\int_{|z|=1} f(z) dz, \quad \int_{|z|=5} f(z) dz, \quad \text{and} \quad \int_{|z-4|=2} f(z) dz$$

where all contours are oriented positively.

Problem 2. The Cauchy distribution is a probability distribution which appears in a variety of contexts, e.g. when one considers random rotations of a line in a plane. It is given by the probability density function $f : \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

Use the residue theorem, a semicircular contour, and a limiting argument to verify that this is a probability density, i.e. that $\int_{\mathbb{R}} f(x) dx = 1$.

Problem 3. Compute the following integrals via contour integration.

- (a) $\int_0^{2\pi} \frac{1}{5-3\cos\theta} d\theta.$
- (b) $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta.$
- (c) $\int_0^\infty \frac{1}{(4x^2+1)^3} dx.$
- (d) $\int_0^\infty \frac{x^2}{(x^2+4)(x^2+9)} dx.$
- (e) $\int_{-\infty}^\infty \frac{x \sin x}{x^2+4x+5} dx.$
- (f) $\int_0^\infty \frac{\cos x}{(1+9x^2)^2} dx.$
- (g) $\int_0^\infty \frac{1}{1+x^6} dx.$
- (h) Boas problem 14.7.37a)

Problem 4. (*Fourier Series*)

Let $f(z) = \sum_{n=0}^\infty a_n z^n$ converge for $|z| < R$.

(a) If $0 < r < R$, show that $f(z) = \sum_{n=0}^\infty a_n r^n e^{in\theta}$ where $z = re^{i\theta}$ and

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta.$$

This is the *Fourier series* of f . *Hint: use Cauchy's integral formula for a_n .*

- (b) Prove Parseval's theorem

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

Hint: expand $|f|^2 = f\bar{f}$ in a series, then integrate term-by-term.

Problem 5. (Argument principle)

Let f be meromorphic on a region Ω . Consider the “logarithmic derivative” $F(z) = f'(z)/f(z)$.

- (a) Let z_0 be an n^{th} order zero of $f(z)$. Show that the residue of F at z_0 is n . (*Hint: use the local description of zeros, i.e. if z_0 is a zero of order n , then in a neighborhood of z_0 the Taylor expansion of f reads*

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \dots$$

- (b) Show that the residue of F at a pole z_0 of order p of $f(z)$ is $-p$ (*Hint: use the local description of a meromorphic function near a pole.*)
 (c) Use the residue theorem and parts (a) and (b) to conclude the argument principle:

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = N - P,$$

where N is the number of zeros of $f(z)$ encircled by γ (counting multiplicity), and P is the number of poles of $f(z)$ encircled by γ (counting multiplicity).

We can use arguments similar to that of the residue theorem to now find zeros and poles! This is extremely useful in control theory and circuit design, among other applications.